SECTION 3.9: RELATED RATES

RECALL: Geometrically, derivatives are slopes of tangent lines. More generally, derivatives are rates of change.

In this section, we usually have two or more related quantities changing with time. Our goal is to relate said quantities then implicitly differentiate with respect to time to relate the rates of said quantities.

GUIDELINES FOR SOLVING RELATED RATE PROBLEMS:

While there is no algorithm which will exactly fit *each and every* application problem, the following guidelines are often helpful to have in the back of your mind when studying related rates:

1. Identify the quantities changing with time and assign them variable names.
2. Use a formula from geometry to relate the variables you listed in Step 1. (A diagram is often helpful!) Check your units at this point to make sure the formula makes sense.
3. Differentiate your formula from Step 2 implicitly with respect to time - this is what 'relates the rates.' Once again, it may be prudent to make sure the units of both sides of your equation match at this point.
4. Substitute the data given in the problem into the related rate equation from Step 3 and solve for the desired rate. Interpret your answer to make sure what you found matches what is asked for in the problem.

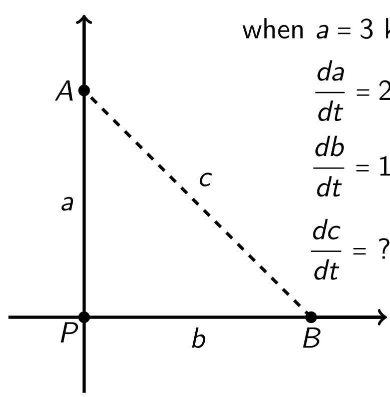
   HINT: One of the most common mistakes is substituting the data into the equations too early. If a quantity is changing with time, wait until this step to substitute it into the equation!

EXAMPLE 1: A road running north to south crosses a road going east to west at the point P. Cyclist A is riding north along the first road, and cyclist B is riding east along the second road. At a particular time, cyclist A is 3 kilometers to the north of P and traveling at 20 km/hr, while cyclist B is 4 kilometers to the east of P and traveling at 15 km/hr. How fast is the distance between the two cyclists changing at that time?

First, we **introduce the variables** a, b, and c, denoting the distance of cyclist A from the point P, the distance of cyclist B from the point P, and the distance between the two cyclists, in that order.

We **identify** the given rates $\frac{da}{dt} = 20$ km/hr, $\frac{db}{dt} = 15$ km/hr, when $a = 3, b = 4$ and the unknown rate $\frac{dc}{dt}$.

Schematically:



when $a = 3$ km, $b = 4$ km

$$\frac{da}{dt} = 20 \text{ km/hr}$$

$$\frac{db}{dt} = 15 \text{ km/hr}$$

$$\frac{dc}{dt} = ?$$

We **find equations** relating the variables a, b, and c. By the Pythagorean Theorem, we always have: $a^2 + b^2 = c^2$. Note here that a, b, and c are all measured in km, so the units of both sides of the equation are $(\text{km})^2$.

Since a, b, and c are functions of time, we **differentiate** both sides of this equation with respect to t.

$$
\begin{aligned}
a^2 + b^2 &= c^2 \\
D_t\left[a^2 + b^2\right] &= D_t\left[c^2\right] \\
D_t\left[a^2\right] + D_t\left[b^2\right] &= D_t\left[c^2\right] \\
2a\frac{da}{dt} + 2b\frac{db}{dt} &= 2c\frac{dc}{dt}
\end{aligned}
$$

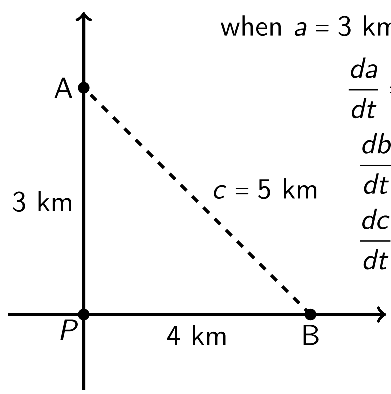
Checking units we note that a, b, and c are all measured in km, so $\frac{da}{dt}$, $\frac{db}{dt}$, and $\frac{dc}{dt}$ are all measured in km/hr.

This means both sides of the related rate equation have units $(\text{km})^2/\text{hr}$.

At the instant $a = 3$ km and $b = 4$ km, we are told $\frac{da}{dt} = 20$ km/hr and $\frac{db}{dt} = 15$ km/hr.

From the Pythagorean Theorem, we know when $a = 3$ km and $b = 4$ km that $c^2 = (3)^2 + (4)^2$, so $c = 5$ km.

Hence, taking a 'snapshot' at the instant when $a = 3$ km and $b = 4$ km, we have:



when $a = 3$ km, and $b = 4$ km, $c = 5$ km

$$\frac{da}{dt} = 20 \text{ km/hr}$$

$$\frac{db}{dt} = 15 \text{ km/hr}$$

$$\frac{dc}{dt} = ?$$

Feeding all of this data into our related rate equation gives:

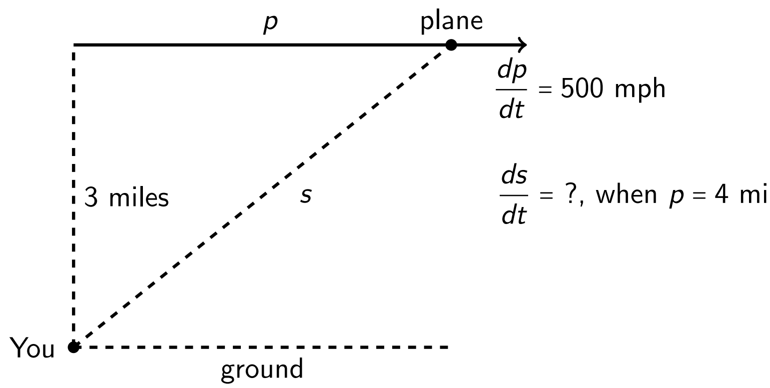
$$
\begin{aligned}
2a\frac{da}{dt} + 2b\frac{db}{dt} &= 2c\frac{dc}{dt} \\
2(3\,\text{km})\left(20\,\frac{\text{km}}{\text{hr}}\right) + 2(4\,\text{km})\left(15\,\frac{\text{km}}{\text{hr}}\right) &= 2(5\,\text{km})\frac{dc}{dt} \\
120\,\frac{\text{km}^2}{\text{hr}} + 120\,\frac{\text{km}^2}{\text{hr}} &= (10\,\text{km})\frac{dc}{dt} \\
240\,\frac{\text{km}^2}{\text{hr}} &= (10\,\text{km})\frac{dc}{dt} \\
\frac{dc}{dt} &= \frac{240}{10}\,\frac{\text{km}^2/\text{hr}}{\text{km}} \\
\frac{dc}{dt} &= 24\,\frac{\text{km}}{\text{hr}}
\end{aligned}
$$

Hence, the distance between the cyclists is increasing at a rate of 24 km/hr at this instant.

EXAMPLE 2: (VIDEO) A plane is flying at an altitude of 3 miles directly away from you at 500 mph. How fast is the plane's distance from you increasing when the plane is flying over a point on the ground 4 miles away?

First, we **introduce variables** p, the distance the plane has traveled after it flew right above you, and s, the distance between you and the plane.

The given rate is $\frac{dp}{dt} = 500$ mph and the unknown rate is $\frac{ds}{dt}$, when $p = 4$. Schematically:



Next we **find equations** relating the variables p and s. By the Pythagorean Theorem we know that at all times:

$$s^2 = p^2 + 9.$$

Next, we **differentiate** both sides of the equation with respect to time:

$$\begin{aligned} s^2 &= p^2 + 9 \\ D_t\left[s^2\right] &= D_t\left[p^2 + 9\right] \\ 2s\,\frac{ds}{dt} &= 2p\,\frac{dp}{dt} \\ s\,\frac{ds}{dt} &= p\,\frac{dp}{dt} \end{aligned}$$

At the instant when $p = 4$ mi, we are told that $\frac{dp}{dt} = 500$ mph. From $s^2 = p^2 + 9$, when $p = 4$ mi, $s = 5$ mi. Feeding all of this into our related rate equation, we get:

$$\begin{aligned} s\,\frac{ds}{dt} &= p\,\frac{dp}{dt} \\ (5\text{mi})\,\frac{ds}{dt} &= (4\,\text{mi})\left(500\,\frac{\text{mi}}{\text{hr}}\right) \\ (5\text{mi})\,\frac{ds}{dt} &= 2000\,\frac{\text{mi}^2}{\text{hr}} \\ \frac{ds}{dt} &= \frac{2000}{5}\,\frac{\text{mi}^2/\text{ hr}}{\text{mi}} \\ \frac{ds}{dt} &= 400\,\frac{\text{mi}}{\text{hr}} \end{aligned}$$

Hence, the plane is moving away at a speed of 400 mph at this instant.

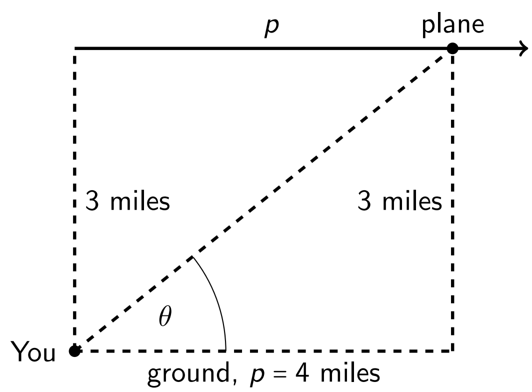
EXAMPLE 3: A plane is flying at an altitude of 3 miles directly away from you at 500 mph . Let θ be the **angle of elevation** of the plane, i.e., the angle between the ground and your line of sight to the plane. How fast (in radians per second) is the angle θ changing when the plane is flying over a point on the ground 4 miles away?

As with the previous example, we let p be the distance the plane has traveled after it flew right above you.

So, the given rate is $\dfrac{dp}{dt} = 500$ mph and the unknown (related) rate is $\dfrac{d\theta}{dt}$, at the instant when $p = 4$ miles.
Since we're told we need to measure the latter in radians per **second**, we convert the units of the given rate, miles per hour, into miles per second:

$$\frac{dp}{dt} = 500\,\frac{\text{miles}}{\text{hour}} = 500\,\frac{\text{miles}}{\text{hour}}\,\frac{1\,\text{hour}}{3600\,\text{seconds}} = \frac{500}{3600}\,\frac{\text{miles}}{\text{second}} = \frac{5}{36}\,\frac{\text{miles}}{\text{second}}$$

Schematically, we have:



$$\frac{dp}{dt} = \frac{5}{36}\text{mi/s}$$

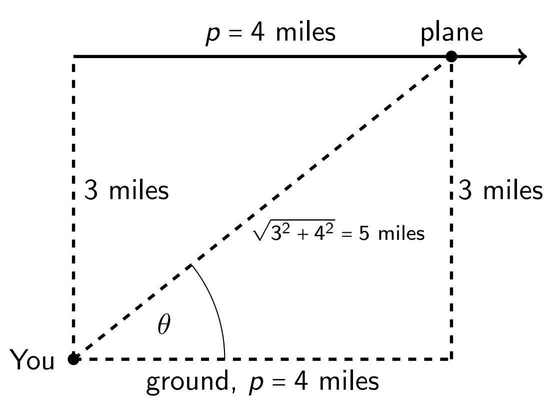
$$\frac{d\theta}{dt} = ?, \text{ when } p = 4 \text{ mi}$$

Now we **find an equation** relating variables p and θ. From the diagram, we see $\tan(\theta) = \dfrac{3\,\text{miles}}{p\,\text{miles}} = 3p^{-1}$

Note the units, $\dfrac{\text{miles}}{\text{miles}}$ cancel out, which makes sense since $\tan(\theta)$ is dimensionless.
Next, we **differentiate** both sides of the equation with respect to time:

$$\begin{aligned}
\tan(\theta) &= 3p^{-1} \\
D_t\left[\tan(\theta)\right] &= D_t\left[3p^{-1}\right] \\
\sec^2(\theta)\,\frac{d\theta}{dt} &= 3(-1)p^{-2}\,\frac{dp}{dt} \\
\sec^2(\theta)\,\frac{d\theta}{dt} &= -3p^{-2}\,\frac{dp}{dt}
\end{aligned}$$

Note when $p = 4$ miles, $\tan(\theta) = \dfrac{3}{4}$ we can use the diagram to get $\sec(\theta) = \dfrac{5}{4}$:



Substituting $\sec(\theta) = \frac{5}{4}$, $p = 4$ miles, and $\frac{dp}{dt} = \frac{5}{36}\frac{\text{miles}}{\text{second}}$ into the related rate equation gives:

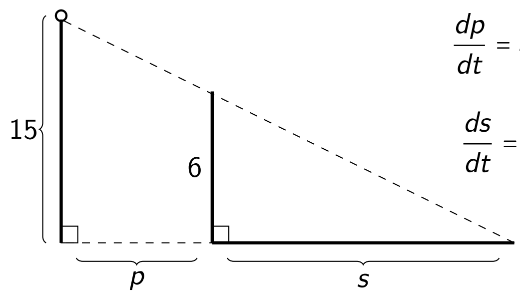
$$\begin{aligned}
\sec^2(\theta)\frac{d\theta}{dt} &= -3p^{-2}\frac{dp}{dt} \\
\left(\frac{5}{4}\right)^2\frac{d\theta}{dt} &= -(3\,\text{miles})(4\,\text{miles})^{-2}\,\frac{5}{36}\,\frac{\text{miles}}{\text{second}} \\
\frac{25}{16}\frac{d\theta}{dt} &= -\frac{3}{16}\,\frac{\text{miles}}{\text{miles}^2}\,\frac{5}{36}\,\frac{\text{miles}}{\text{second}} \\
\frac{25}{16}\frac{d\theta}{dt} &= -\frac{15}{576}\,\frac{\text{miles}^2}{\text{miles}^2\,\text{second}} \\
\frac{d\theta}{dt} &= -\frac{16}{25}\,\frac{15}{576}\,\frac{1}{\text{second}} \\
\frac{d\theta}{dt} &= -\frac{1}{60}\,\frac{1}{\text{second}}
\end{aligned}$$

Recalling that radians are dimensionless quantities, we interpret $-\frac{1}{60}\frac{1}{\text{second}} = -\frac{1}{60}\frac{\text{radian}}{\text{second}}$.

Hence, the angle of inclination is decreasing at a rate of 1 radian per second at this instant.

EXAMPLE 4: (VIDEO) It is night. Someone who is 6 feet tall is walking away from a street light at a rate of 3 feet per second. The street light is 15 feet tall. The person casts a shadow on the ground in front of them. How fast is the length of the shadow growing when the person is 7 feet from the street light? How fast is the tip of the shadow moving at this instant?

First, we **introduce two variables**, p, the distance from the person to the lamp, and s, the length of the shadow. The given rate is $\dfrac{dp}{dt} = 3$ ft/s and the unknown (related) rate is $\dfrac{ds}{dt}$ at the moment when $p = 7$ feet.



$$\frac{dp}{dt} = 3\text{ft/s}$$

$$\frac{ds}{dt} = ?,\ \text{when } p = 7\text{ft}$$

Now we **find equations** that relate the variables p and s. We use the fact that we have similar triangles to write:

$$\begin{aligned} \frac{s+p}{15} &= \frac{s}{6} \\ 6(s+p) &= 15s \\ 6s + 6p &= 15s \\ 6p &= 9s \\ 2p &= 3s \end{aligned}$$

Next, we **differentiate** both sides of the equation above with respect to time:

$$\begin{aligned} D_t[2p] &= D_t[3s] \\ 2\frac{dp}{dt} &= 3\frac{ds}{dt} \end{aligned}$$

When $p = 7$ ft, we substitute $\dfrac{dp}{dt} = 3$ ft/s:

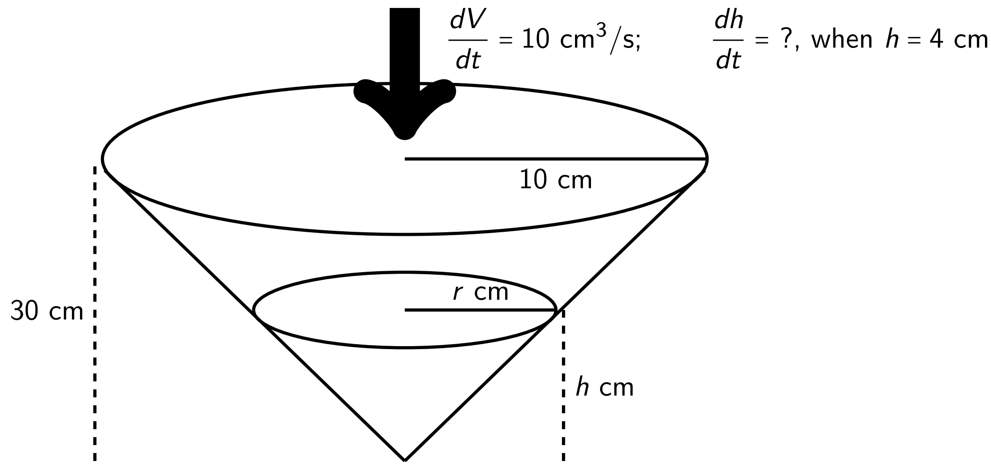
$$\begin{aligned} 2\frac{dp}{dt} &= 3\frac{ds}{dt} \\ 2\left(3\,\frac{\text{ft}}{\text{s}}\right) &= 3\frac{ds}{dt} \\ 6\,\frac{\text{ft}}{\text{s}} &= 3\frac{ds}{dt} \\ \frac{ds}{dt} &= 2\,\frac{\text{ft}}{\text{s}} \end{aligned}$$

Therefore, the shadow is growing at a rate of 2 ft/s when the person is 7 ft from the lamp. Since the person is walking at a rate of 3 ft/s, the tip of the shadow is moving at a rate of 3 ft/s + 2 ft/s = 5 ft/s.

EXAMPLE 5: Water is poured into a conical container at the rate of 10 cm^3/s (cubic centimeters per second.) The cone points directly down, and it has a height of 30 cm and a base radius of 10 cm. How fast is the water level rising when the water is 4 cm deep?

First, we **introduce several variables**. Let V be the volume of the water in the container, let r be the radius of the circular surface of the water, and let h be the depth of the water in the container.

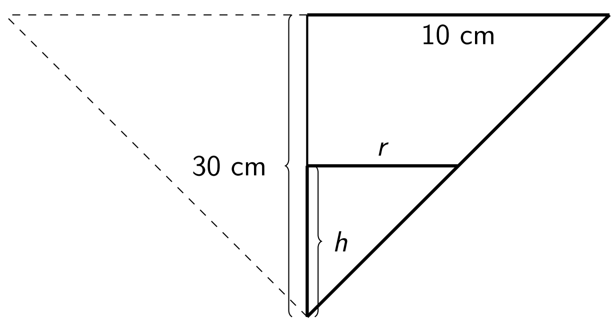
The given rate is $\frac{dV}{dt} = 10\ cm^3/s$, and the unknown rate is $\frac{dh}{dt}$, at the moment when $h = 4$ cm. Schematically:



Now we **find equations** that relate all the variables. Notice that water in the container assumes the shape of the container. In this example, the shape is a cone. Therefore, we use the formula for the volume of a cone

$$V = \frac{\pi}{3} r^2 h.$$

Let's draw a cross section of the cone.



Notice a big right triangle and a smaller right triangle inside the big one. These two right triangles are similar, because their corresponding angles are equal. Since the ratios of corresponding sides in similar triangles are equal:

$$\frac{r}{h} = \frac{10}{30} \qquad \text{so} \qquad r = \frac{h}{3}.$$

At this point we could **differentiate** both sides of each equation, but we can take a simpler approach by combining these two equations and expressing V in terms of h:

$$\begin{aligned} V &= \frac{\pi}{3} r^2 h \\ V &= \frac{\pi}{3} \left(\frac{h}{3}\right)^2 h \quad \text{Since } r = \frac{h}{3} \\ V &= \frac{\pi}{27} h^3 \end{aligned}$$

Now, we **differentiate** with respect to time:

$$\begin{aligned}
V &= \frac{\pi}{27}h^3 \\
D_t[V] &= D_t\left[\frac{\pi}{27}h^3\right] \\
\frac{dV}{dt} &= \frac{\pi}{27}D_t\left[h^3\right] \\
\frac{dV}{dt} &= \frac{\pi}{27}(3h^2)\,\frac{dh}{dt} \\
\frac{dV}{dt} &= \frac{\pi}{9}\,h^2\,\frac{dh}{dt}
\end{aligned}$$

Substituting values for $\frac{dV}{dt}$, h, and $\frac{dh}{dt}$ when $h = 4$ cm gives:

$$\begin{aligned}
\frac{dV}{dt} &= \frac{\pi}{9}\,h^2\,\frac{dh}{dt} \\
10\,\frac{\text{cm}^3}{\text{s}} &= \frac{\pi}{9}\,(4\,\text{cm})^2\,\frac{dh}{dt} \\
10\,\frac{\text{cm}^3}{\text{s}} &= \frac{16\pi}{9}\,\text{cm}^2\,\frac{dh}{dt} \\
\frac{dh}{dt} &= 10\,\frac{\text{cm}^3}{\text{s}}\,\frac{9}{16\pi}\,\frac{1}{\text{cm}^2} \\
\frac{dh}{dt} &= \frac{90}{16\pi}\,\frac{\text{cm}^2}{\text{cm}^2\,\text{s}} \\
\frac{dh}{dt} &= \frac{45}{8\pi}\,\frac{\text{cm}}{\text{s}}
\end{aligned}$$

At the instant the water in the container is 4 cm deep, the water level is rising at the rate of $\frac{45}{8\pi}$ cm/s.

HOMEWORK: Section 3.9: 11 - 51 every other odd, 53*, 57*, 61*